

Canard cycles in generic slow-fast systems on the two-torus

How many ducks can dance on the torus?

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Definition

- Consider **planar slow-fast system** of the following form:

$$\begin{cases} \dot{x} = f(x, y, \varepsilon), \\ \dot{y} = \varepsilon g(x, y, \varepsilon), \end{cases} \quad \varepsilon \in (\mathbb{R}, 0). \quad (1)$$

- Variables: x is a fast variable, and y is a slow one, ε is a small parameter.
- Slow curve is a set $M := \{(x, y) \mid f(x, y, 0) = 0\}$.
- Duck* (or *canard*) solutions are solutions, whose phase curves contain an arc of length bounded away from 0 uniformly in ε , that keeps close to the *unstable* part of the slow curve
- Canard cycle* is a limit cycle which is a canard.

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Ducks on the torus: introduction

- Consider slow-fast system on the **two-torus**
- Pick a point far from M
- Consider its trajectory in forward time
- Reverse the time
- When ε decreases, L moves down, and R moves up.
- For some ε , we've got canard cycle.

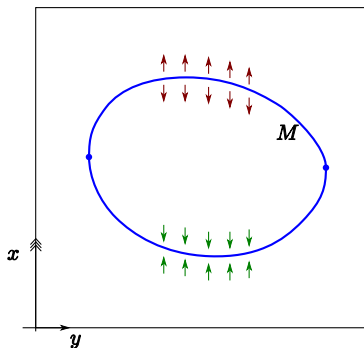


Figure: Ducks on the torus
(Yu. S. Ilyashenko, J. Guckenheimer, 2001)

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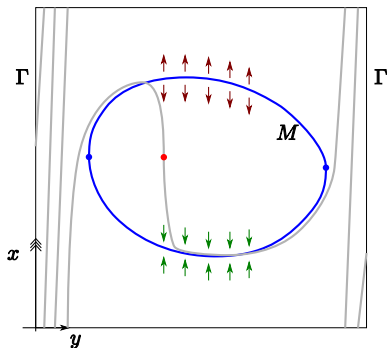


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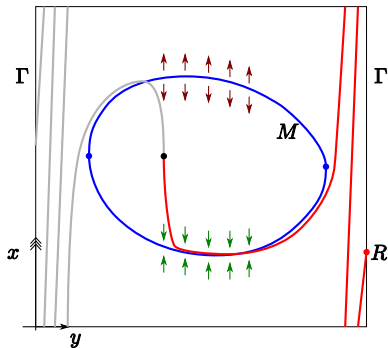


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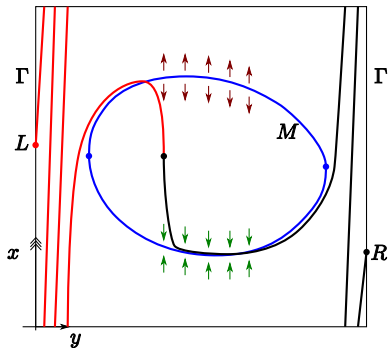


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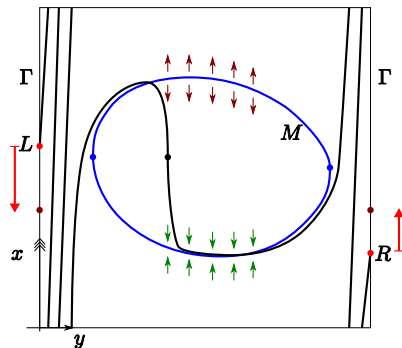


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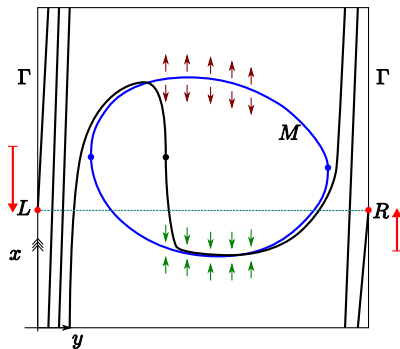


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Main results: the structure of ducky area

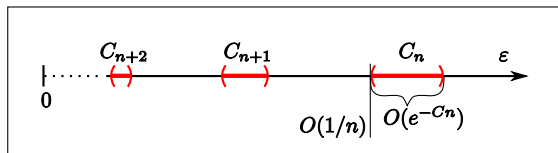


Figure: Intervals C_n : the ducks live here

- There exists a sequence of intervals $\{C_n\}_{n=1}^{\infty}$ such that for every $\varepsilon \in C_n$ the system has *attracting* canard cycles.
- Intervals C_n are exponentially small.
- They accumulate to 0.
- Their density is 0 near $\varepsilon = 0$.
- The number of canard cycles is bounded by the number of folds of M .

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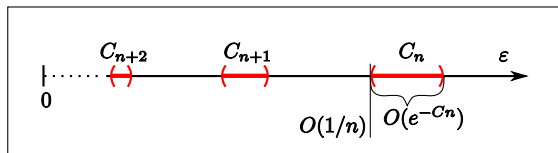


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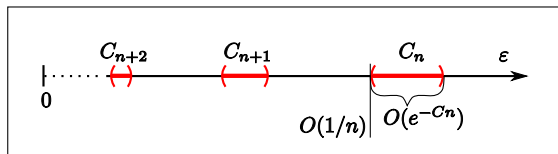


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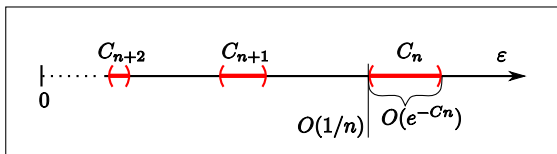


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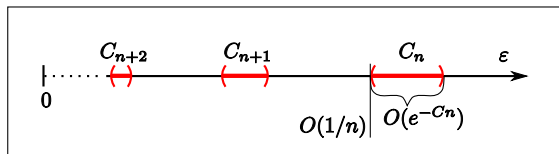


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How many ducks can dance on the torus?

Theorem 1 (Upper estimate for the number of canards)

Consider slow-fast system on the two-torus, i.e. $(x, y) \in \mathbb{T}^2$, and the speed of the slow motion is bounded away from zero ($g > 0$).

Assume M is connected nondegenerate curve with $2N$ fold points, $N < \infty$, and some additional nondegeneracy assumptions hold.

Then there exists number $0 < K \leq N$, such that the following assertions hold:

- There exists a sequence $\{C_n\}_{n=1}^{\infty}$ of intervals on the ray $\{\varepsilon > 0\}$, accumulating to 0, such that for every $\varepsilon \in C_n$, the system has exactly $2K$ canard cycles (K attracting and K repelling).*
- For any $\varepsilon > 0$ small enough, the number of limit cycles that make one rotation along y -axis is bounded by $2K$.*
- Their basins have bounded away from 0 measure.*

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How many ducks can dance on the torus? (2)

Remark

*It follows from theorem 1, that for convex slow curve, there exists **exactly one** pair of canard cycles*

Remark

The number K of canard cycles can be effectively computed without intergration of the system.

Theorem 2 (Sharp estimate for K)

For every $N > 0$ there exists an open set in the space of slow-fast systems on the two-torus for which the number of canard cycles reaches its maximum: $K = N$.

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Sketch of the proof: Poincaré map

- Consider Poincaré map $P : \Gamma \rightarrow \Gamma$.
- Use Rolle's lemma to estimate number of fixed points.
- Fixed points located near roots of $P'(x) = 1$ correspond to canard cycles.

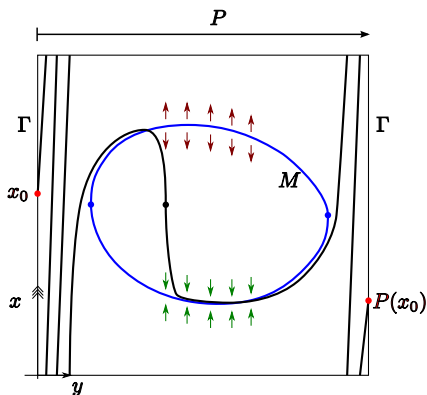


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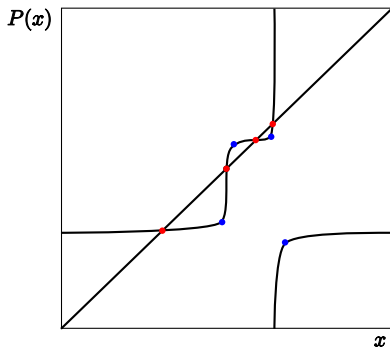


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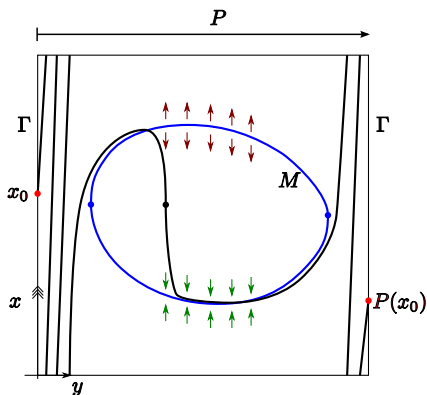


Figure: Poincaré map

Main technical result

- Consider interval J and Poincaré map $Q : J \rightarrow \Gamma$.
- $\log Q'(u) = \frac{1}{\varepsilon} \int_{FG} f'_x(x, y, 0) dx + o\left(\frac{1}{\varepsilon}\right)$
- Main obstacle: dynamics near the jump point.
- Solution: Distortion Lemma (Denjoy-Schwartz).

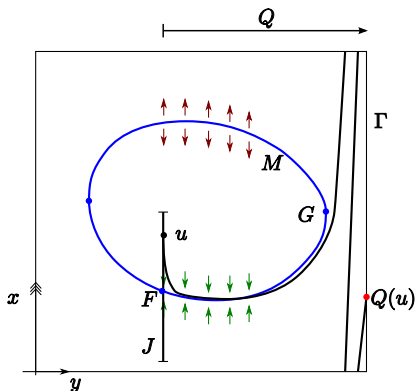


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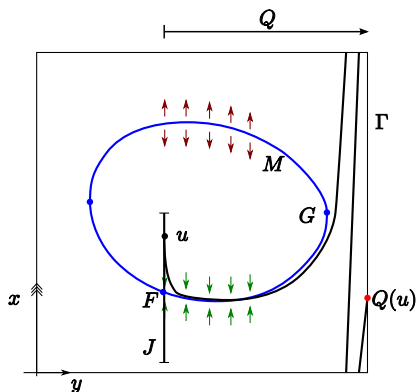


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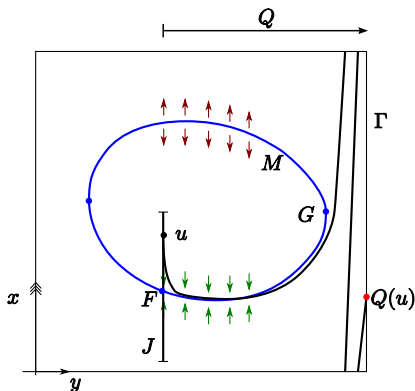


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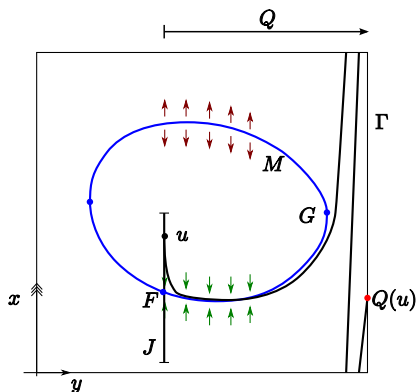


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The duck farm

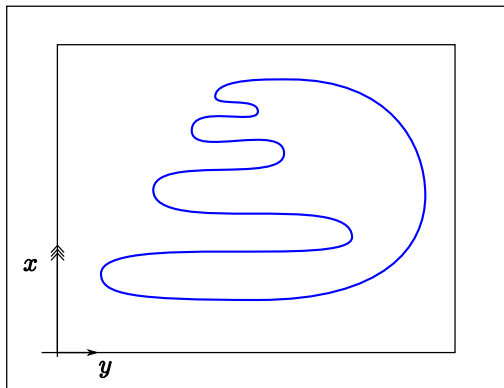


Figure: The construction of open set of slow-fast systems with maximal number of canard cycles. E.g. $K = N = 4$ on the figure.

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