

# Canard cycles in generic slow-fast systems on the two-torus

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## Abstract

We show that there exist generic slow-fast systems with only one (time-scaling) parameter on the two-torus, which have attracting canard cycles for arbitrary small values of this parameter. This is in drastic contrast with the planar case, where canards usually occur in two-parametric families. The number of canard cycles is no more than the number of fold points of the slow curve. This estimate is exact for every system from some open set.

**Introduction.** Consider a generic slow-fast system:

$$\begin{cases} \dot{x} = f(x, y, \varepsilon) \\ \dot{y} = \varepsilon g(x, y, \varepsilon) \end{cases} \quad \varepsilon \in (\mathbb{R}, 0). \quad (1)$$

For the planar case (e.g.  $(x, y) \in \mathbb{R}^2$ ), there is a rather simple description of its behavior for small  $\varepsilon$ . It consists of interchanging phases of slow motion along stable parts of the slow curve  $M := \{(x, y) \mid f(x, y, 0) = 0\}$  and fast jumps along straight lines  $y = \text{const}$ . Given additional parameters, which depend on  $\varepsilon$ , one can observe more complicated behavior: appearance of *duck* (or *canard*) solutions (particularly limit cycles), i.e. solutions, whose phase curves contain an arc of length bounded away from 0 uniformly in  $\varepsilon$ , that keeps close to the unstable part of the slow curve.

In [GI], Yu. S. Ilyashenko and J. Guckenheimer discovered a new kind of behavior of slow-fast systems on the two-torus. It was shown that for some particular family with no auxiliary parameters there exists a sequence

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of intervals accumulating at 0, such that for any  $\varepsilon$  from these intervals, the system has exactly two limit cycles, both of which are canards, where one is stable and the other unstable. Yu. S. Ilyashenko and J. Guckenheimer conjectured that there exists an open domain in the space of slow-fast systems on the two-torus with the same property. Here we present proof of this conjecture.

**Results.** Consider the system (1) and assume that the phase space is the two-torus:

$$(x, y) \in \mathbb{T}^2 \cong \mathbb{R}^2 / (2\pi\mathbb{Z}^2). \quad (2)$$

Assume that the speed of the slow motion is bounded away from zero ( $g > 0$ ), the slow curve  $M$  is a smooth connected curve, and its lift to the covering coordinate plane is contained in the interior of the fundamental square  $\{|x| < \pi, |y| < \pi\}$ . We also assume that all fold points (i.e. the points of  $M$  where the tangent to  $M$  is parallel to  $x$ -axis) are nondegenerate (the tangency rate is quadratic). In this case, the number of fold points is finite and even: let us denote it by  $2N$ .

**Theorem 1.** *For any generic slow-fast system on the two-torus with the described properties, under some additional nondegeneracy assumptions, the following properties hold. There exists a positive number  $k \leq N$  and a sequence of intervals accumulating to zero, such that for every  $\varepsilon$  that belongs to these intervals the system has exactly  $k$  attracting and  $k$  repelling limit cycles, which make one rotation along  $y$ -axis during the period. All these cycles are canards. The measure of their basins is bounded from below uniformly for  $\varepsilon \rightarrow 0^+$ . For any small  $\varepsilon > 0$ , the number of limit cycles that make one rotation along  $y$ -axis is bounded by  $2k$ .*

**Theorem 2.** *There exists an open set in the space of slow-fast systems on the two-torus for which the number  $k$  of attracting canard cycles is maximal and equal to  $N$ .*

The presented proof of these theorems uses well-developed techniques of geometric singular perturbation theory as well as recent results in the theory of normal forms and new technical results on the slow-fast dynamics on the two-torus.

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## References

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